["Stochastic Modelling for System Biology" Chapter 5]

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Definitions

Definition 1 (Markov Chain)

S : state-space(countable set), θ : stochastic variable,
$\{\theta^{(t)} \in S | t = 0, 1, ...\}$: discrete time stochastic processes
If $\{\theta^{(t)}\}_t$ satisfies

$$P(\theta^{(t+1)} \in A | \theta^{(0)} = x_0, ..., \theta^{(t)} = x) = P(\theta^{(t+1)} \in A | \theta^{(t)} = x)$$

$$\forall x, x_{t-1}, ..., x_0 \in S, A \subset S, t = 0, 1, ...$$

then, $\{\theta^{(t)}\}_t$ is said to be **Markov Chain**.

The future states only depend on present states, not on the past states.

frame title

Definition 2 (time-homogeneous)

if $\{\theta^{(t)}\}_t$ satisfies

$$P(\theta^{(t+1)} \in A|\theta^{(t)} = x) = P(\theta^{(t+1)} \in A|\theta^{(t)} = x) := P(x|A),$$

$\forall x, x_{t-1}, ..., x_0 \in S, A \subset S, t = 0, 1, ...$
Then, Markov Chain is said to be **time-homogeneous**.

States transition probabilities are independent from time, t.

Stochastic matrix

Definition 3

Define $P(x, y)$ as below.

$$P(x, y) := P(\theta^{(t+1)} = y | \theta^{(t)} = x)$$

Then, **stochastic matrix** P is defined as below.

$$P := \begin{pmatrix} P(x_1, x_1) & \cdots & P(x_1, x_r) \\ \vdots & \ddots & \vdots \\ P(x_r, x_1) & \cdots & P(x_r, x_r) \end{pmatrix}$$

More generally, A real $r * r$ matrix P is said to be **stochastic matrix** if and only if its elements are all non-negative and its rows sums to 1.

Examples

An example of a naive Markov Chain.

Let's look at the situation shown on the right.

$$P(\theta^{(t+1)} = 1 | \theta^{(t)} = 1) = 0.6$$

$$P(\theta^{(t+1)} = 1 | \theta^{(t)} = 2) = 0.2$$

$$P(\theta^{(t+1)} = 2 | \theta^{(t)} = 1) = 0.4$$

$$\vdots$$

Each probability is independent from t.
The transition matrix P is below.

$$P := \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.2 & 0.3 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}$$

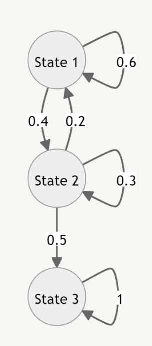

Figure 1:

Lemmas

Lemma 4

The product of two stochastic matrices is another stochastic matrix.

Proof.
let $P = (p_{ij})_{ij}, Q = (q_{ij})_{ij}$ be stochastic matrices. From definition of matrix product, $(PQ)_{ij} = \sum_k p_{ik} q_{kj}$. Also, stochastic matrices satisfies $\forall i \in \mathbb{N}, \quad \sum_j p_{ij} = 1, \quad \sum_j q_{ij} = 1$

$$\therefore \sum_j (PQ)_{ij} = \sum_j \sum_k p_{ik} q_{kj} = \sum_k \left(p_{ik} \sum_j q_{kj} \right) = \sum_k p_{ik} = 1$$

Therefore, sum of ith row of PQ equals to 1, thus PQ is another stochastic matrix. □

Lemma 5

Every eigenvalue λ satisfies $|\lambda| \leq 1$.

Proof.
Let $P = (p_{ij})_{ij}$ be stochastic matrices, $v = (v_i)_i$ be eigenvector of P, and λ be eigenvalue.
Also, define m as below.

$$m =_i \{|v_1|, \ldots, |v_n|\}$$

Then, because $Pv = \lambda v$,

$$\sum_i p_{mi} v_i = \lambda v_m$$

$$\therefore |\lambda| = \frac{|\sum_i p_{mi} v_i|}{|v_m|} \leq \frac{\sum_i p_{mi} |v_i|}{|v_m|} \leq \frac{\sum_i p_{mi} |v_m|}{|v_m|} = \frac{1 \cdot |v_m|}{|v_m|} = 1 \quad \square$$

Lemma 6

Every stochastic matrix has at least one eigenvalue equal to 1.

Proof.

Let $P = (p_{ij})_{ij}$ be a stochastic matrix, v_0 be a vector s.t.$v_0 = (1, \cdots, 1)$, then

$$\forall i, (Pv_0)_i = \sum_j p_{ij} = 1 = (v_0)_i \therefore Pv_0 = v_0$$

Thus, P has at least one eigenvalue equal to 1. □

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What we will learn next

1. **Discrete** State-space -> **Continuous** state-space
   1. ex) Processes with Gaussian Noise like AR(1):

$$Z_{t+1} = \alpha Z_t + \varepsilon_t, \quad t = 0, 1, \ldots, \quad \varepsilon \sim N(0, \sigma^2)$$

2. **Dicrete** time -> **Continuous** time
   1. ex) Random Walk -> Brownian Motion

I will introduce further theories at next b-seminar!

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Thank You!

References I

[Wil19] Darren J. Wilkinson. *Stochastic Modelling for Systems Biology, Third Edition*. Chapman Hall, 2019. ISBN: 9780367656935. URL: https://www.routledge.com/Stochastic-Modelling-for-Systems-Biology-Third-Edition/Wilkinson/p/book/9780367656935.