the relationship between the Euclid's algorithm and the number of digits.

~ focusing on the Fibonacci sequence ~

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Abstract

the Euclid's algorithm is a method for computing the greatest common divisor of two positive integers a and b with $a > b$ by continuous division. This study introduces how large the number of steps will be depend on a and b. Also, This introduces the maximum value of x and then a and b and the limit value of x when a and b are consecutive Fibonacci numbers. (x is defined as a ratio of the number of digits of b to the number of steps.)

Introduction

How large will the number of steps be? The larger a and b are, the more the number of steps will be. Thus it is no use just finding the maximum value of the number of steps. That's why x is defined as a ratio of the number of digits of b to the number of steps.

Definitions of terms

the number of steps …How many times the numbers are divided in the Euclid's algorithm.

$\boldsymbol{x}$… a ratio of the number of digits of b to the number of steps.

$$x = \frac{\text{the number of steps.}}{\text{the number of digits of } b}$$

$\boldsymbol{F(a,b)}$…the number of steps needed by the Euclid's algorithm to process a and b.

The purpose of my study

To find the maximum value of x and then a, b. In other word, to find the maximum value of the ratio between the number of steps and the number of digits of b.

The relationship between the Euclid's algorithm and the Fibonacci series

$$a = q_1 * b + r_1$$
$$b = q_2 * r_1 + r_2$$
$$r_1 = q_3 * r_2 + r_3$$
$$\vdots$$
$$r_{n-3} = q_{n-1} * r_{n-2} + r_{n-1}$$
$$r_{n-2} = q_n * r_{n-1} + r_n$$
$$r_{n-1} = q_{n+1} * r_n + 0$$

General form of the Euclid's algorithm

JUST THE SAME!
IF $q_n = 1$

$$f_{n+1} = f_n + f_{n+1}$$
$$f_n = f_{n-1} + f_{n-2}$$
$$f_{n-1} = f_{n-2} + f_{n-3}$$
$$\vdots$$
$$f_5 = f_4 + f_3$$
$$f_4 = f_3 + f_2$$
$$f_3 = f_2 + f_1$$

General form of the Fibonacci series

In the algorithm, the smaller a decreasing rate of $r_k = 1$ $(k = 1,2,3,,,n)$is, the smaller the steps will be. Therefore steps will be larger if $q_k = 1$ $(k = 1,2,3,,,n)$. Then it is easy to see that General form of the algorithm and it of the Fibonacci series are just the same. Thus we'll focus on the of them

Since if $a = f_{n+1}, b = f_n$, the steps will be large, we'll find the maximum value of x by focusing on the series.

Theorem1 Let integers a and b, $a > b > 0$, be the numbers processed by the the Euclid's algorithm, then the number of steps in an application of the algorithm never exceeds 5 times the number of digits of b.Also, all pairs (a, b) such that $x = \frac{F(a,b)}{\text{the digits of } b} = 5$ are only these 3 pairs, $(b, a) = (f_6, f_7), (f_{11}, f_{12}), (f_{16}, f_{17}) = (8,13), (89,144), (987,1597)$

ex)if $\boldsymbol{a = 144, b = 89}$

①$144 = 1 * 89 + 55 = f_{12} = f_{11} + f_{10}$
②$89 = 1 * 55 + 34 = f_{11} = f_{10} + f_9$
③$55 = 1 * 34 + 21 = f_{10} = f_9 + f_8$
④$34 = 1 * 21 + 13 = f_9 = f_8 + f_7$
⑤$21 = 1 * 13 + 8 = f_8 = f_7 + f_6$
⑥$13 = 1 * 8 + 5 = f_7 = f_6 + f_5$
⑦$8 = 1 * 5 + 3 = f_6 = f_5 + f_4$
⑧$5 = 1 * 3 + 2 = f_5 = f_4 + f_3$
⑨$3 = 1 * 2 + 1 = f_4 = f_3 + f_2$
⑩$2 = 2 * 1 + 0 = f_3 = f_2 + f_1$

$\uparrow \gcd(89,144) = 1$

$x = \frac{F(144,89)}{\text{the digits of } b} = 5$

ex)if $\boldsymbol{a = 13, b = 8}$

①$13 = 1 * 8 + 5 = f_7 = f_6 + f_5$
②$8 = 1 * 5 + 3 = f_6 = f_5 + f_4$
③$5 = 1 * 3 + 2 = f_5 = f_4 + f_3$
④$3 = 1 * 2 + 1 = f_4 = f_3 + f_2$
⑤$2 = 2 * 1 + 0 = f_3 = f_2 + f_1$

$\uparrow \gcd(8,13) = 1$

$x = \frac{F(13,8)}{\text{the digits of } b} = 5$

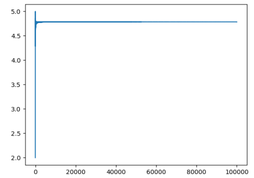
In the process of proving theorem1, it is also proved that $x = \frac{n-1}{\left\lfloor \log_{10}\left\{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right\}\right\rfloor + 1}$ if $a = f_{n+1}, b = f_n$.

Theorem2 Let k be the number of digits of b. The number of steps needed by the Euclid's algorithm to process a and b is at the most $\left\lfloor \frac{k - \log_{10}\frac{1}{\sqrt{5}}}{\log_{10}\left(\frac{1+\sqrt{5}}{2}\right)} \right\rfloor - 1$;However, let $\lfloor r \rfloor$ be the greatest integer less than or equal to r.

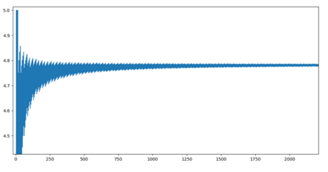
If $k = 2$, in other words,if the number of digits of $b = 2$, from theorem1, since $x = \frac{F(a,b)}{\text{the digits of } b} \leq 5$, the number of steps is not more than 10.
In the same way, If $k = 100$, in other words,if the number of digits of $b = 100$, the number of steps is not more than 500. However, we can more correctly say that the number of steps is not more than $\left\lfloor \frac{100 - \log_{10}\frac{1}{\sqrt{5}}}{\log_{10}\left(\frac{1+\sqrt{5}}{2}\right)} \right\rfloor - 1 = \lfloor 480.1 \rfloor - 1 = 479$ by using theorem2.

Theorem3 $\boldsymbol{x} = \frac{n-1}{\left\lfloor \log_p\left\{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right\}\right\rfloor + 1}$ $(\boldsymbol{n \neq 1, 2, 6, 12})$ and $\lim_{n\to\infty} \frac{n-1}{\left(\left\lfloor \log_p\left\{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right\}\right\rfloor + 1\right)} = \frac{1}{\log_p\left(\frac{1+\sqrt{5}}{2}\right)}$ if $\boldsymbol{a = f_{n+1}, b = f_n, x} = \frac{F(a,b)}{\text{the digits of } b \text{ in the base } m}$.

The following graph1,2 shows that when$a = f_{n+1}, b = f_n$
The vertical line shows n of $b = f_n$, and the horizontal one shows x.



Graph1



Graph2

Future prospect

・To expand the range of a, b from natural numbers to real numbers
・To find the maximum value of x if the numbers are converted to base m.